Spectrally-Constrained Sequences: Bounds and Constructions
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Abstract—We investigate spectrally-constrained sequences (SCSs) which are applicable to communication and radar systems operating over non-contiguous carriers or frequency slots. Typical examples of such systems are overlay cognitive radio or cognitive radar networks. First, we derive the periodic- and aperiodic- correlation lower bounds for single-channel SCSs and multi-channel SCSs by convex optimization in the frequency domain. Each of these bounds reduces to a Welch bound when the number of forbidden carriers is set to zero. We then propose systematic constructions of optimal unimodular single-channel SCSs with the aid of cyclic difference sets and the theory of maximal-length shift register sequences.

Index Terms—Spectrally-Constrained Sequences (SCS), Cognitive Radio, Cognitive Radar, Complementary Sequence Set, Cyclic Difference Set, Maximal-length Shift Register Sequences, Carrier Aggregation.

I. INTRODUCTION

The design of sequences with good correlation properties plays a pivotal role in wireless communications and radar sensing for numerous applications such as channel estimation, synchronization, spread spectrum communications, surveillance and ranging, etc [1]. Traditional sequence design generally assumes the availability of a contiguous spectral band, meaning that the sequence energy can be allocated to all the carriers of such a spectral band. To satisfy this requirement, a simple and commonly used scheme is to assign each application a separate contiguous spectral band with guard bands inserted at both ends so that adjacent bands (that serve different applications) will not overlap. It is noted that perfect sequences with zero periodic autocorrelation sidelobes [2], [3] is possible if and only if a contiguous spectral band is allocated.

Nevertheless, the aforementioned spectrum allocation scheme has become very difficult to continue nowadays because of the increasingly congested and fragmented spectrum. As numerous applications in wireless communications and radar sensing all rely on the finite and precious spectral resource, the research community has begun to rethink how to significantly increase the spectrum utilization in a much more efficient way [4]. A resultant new paradigm is called cognitive radio [5], [6] (or cognitive radar [7]) which refers to a set of intelligent radio devices that are capable of sensing and detecting the RF environment for unused spectral bands and “borrowing” them for temporary information exchange. In particular, an overlay cognitive radio network keeps searching for white space (in some licensed spectrum) to serve new users, called secondary users, while ensuring minimal interference to the licensed users.

In this paper, we consider an overlay cognitive radio (or cognitive radar) network where multiple secondary users are separated by distinct signature sequences and communicate over non-contiguous carriers. Sequences satisfying such constraint in the frequency domain are called spectrally-constrained sequences (SCSs)\(^2\). Two types of SCSs will be considered: multi-channel SCSs and single-channel SCSs. In particular, the former is defined by the correlation sums of at least two constituent sequences and thus is formally referred to as SC complementary sequence sets (SC-SCSs). For more information on perfect- and quasi- complementary sequence sets with no spectral constraints, the reader is referred to [8]–[12].

Traditional sequences in general are not applicable in spectrally-constrained systems. As shown in [13], the correlation properties of a traditional zero-correlation zone (ZCZ) sequence set [14]–[17] will be damaged/lost if spectral nulling of sequences is imposed. A perfect sequence with zero periodic autocorrelation sidelobes is impossible if one or more carriers are nulled. There has been a surge of interest in generating SCSs from a numerical optimization point of view. Most of these works are only concerned with the design of single-channel SCSs. He et al. presented two algorithms in [18] for unimodular SCSs (with low aperiodic autocorrelations) for use in cognitive radar. Low autocorrelation SCSs with low peak-to-average power ratio (PAPR) are reported in [19] by taking advantage of the Gerchberg-Saxton (GS) algorithm [20]. Rowe et al. designed SCSs by developing the so-called SHAPE algorithm based on the Fast Fourier Transform (FFT) [21]. A nonconvex quadratic programming problem for SCS design has been formulated in [22] and solved by relaxing it into a convex optimization problem. Remarkable advances have been made by Song, Bahu, and

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1See Remark 5 in Section III.

2In some literature, an SCS may also be called a cognitive sequence or cognitive waveform.
Palomar by applying majorization-minimization technique to design unimodular single-channel SCSs and multi-channel SCSs in [23] and [24], respectively. Hu et al. proposed a novel two-dimensional time-frequency sequence synthesis for transform domain communication systems (TDCS) in overlay CR networks [25]. In addition, a hybrid approach was developed in [13] to generate “quasi-ZCZ” SCSs with advantages of zero out-of-band power leakage, fast generation, and low computation load.

The principal objective of this paper is to derive the correlation lower bounds of SCSs to measure the optimality of SCSs obtained from various approaches. We note that the well-known Welch bounds [26], Sarwate bounds [27], Levenshtein bounds [28], and Tang-Fan bounds [29] are only applicable to known Welch bounds [26], Sarwate bounds [27], Levenshtein bounds [28], and Tang-Fan bounds [29] are only applicable to traditional sequences with no spectral constraints. Motivated by this, we aim to study generalized bounds with spectral constraints.

Our main contributions are two sets of lower bounds on the periodic- and aperiodic- correlations of SCSs. These bounds are obtained via convex optimization in the frequency domain and are functions of $\rho$, the maximal power leakage into the forbidden carriers (i.e., frequency slots which are occupied by other applications and thus should not be interrupted). When the number of forbidden carriers is zero, meaning that no spectral constraint is imposed, these bounds reduce to the corresponding Welch bounds, respectively. We investigate conditions on how to meet these bounds with equality. In addition, we present systematic constructions of unimodular single-channel SCSs, each having uniformly low periodic autocorrelation sidelobes that meet one of our derived lower bounds with equality.

The rest of this paper organizes as follows. In Section II, we give some necessary notations and definitions including that for spectral constraint, power leakage, CSS, and SCS. We then give a brief introduction to cyclic difference set as it plays an important role in our proposed constructions for single-channel SCSs. In Section III, we derive periodic- and aperiodic- correlation lower bounds for both types of SCSs and show the conditions when these bounds are met with equality. Some bound tightness discussions are first presented in Section IV, followed by our proposed unimodular single-channel SCSs which have uniformly low periodic autocorrelation sidelobes. Finally, this work is concluded in Section V.

II. PRELIMINARIES

A. Notations and Definitions

The following notations will be used throughout this paper.

- Denote by $A^H$ and $A^T$ the conjugate transpose and the transpose of matrix $A$, respectively;
- Denote by $\|A\|_F := \sqrt{\sum_{i=0}^{M-1} \sum_{j=0}^{N-1} |a_{i,j}|^2}$ the Frobenius norm of matrix $A = [a_{i,j}]_{i=0,j=0}^{M-1,N-1}$;
- $\omega_N = \exp(\sqrt{-2\pi i/N})$, the $N$th root of unity;
- Denote by $I_N$ the identity matrix of order $N$;
- Denote by $0^{M \times N}$ the zero matrix of order $M \times N$;
- Denote by $F_N = [f_{i,j}]_{i,j=0}^{N-1}$ the (scaled) discrete Fourier transform (DFT) matrix of order $N$, i.e., $f_{i,j} = \frac{1}{\sqrt{N}} \omega_N^{-ij}$, for $0 \leq i,j \leq N-1$. It is noted that $F_N F_N^H = I_N$.

Given two length-$N$ complex-valued sequences $a$ and $b$, their aperiodic correlation function at time-shift $\tau$ is defined as

$$ R(a, b)(\tau) \triangleq \begin{cases} \sum_{t=0}^{N-1-\tau} a_t b_{t+\tau} & 0 \leq \tau \leq N-1; \\ \sum_{t=-1}^{N-1+\tau} a_{t-\tau} b_t & 1 - N \leq \tau \leq 1; \\ 0 & |\tau| \geq N. \end{cases} $$

(1)

When $a \neq b$, $R(a, b)(\tau)$ is called the aperiodic cross-correlation function (ACCF) between $a$ and $b$; Otherwise, it is called the aperiodic auto-correlation function (AACF) of $a$. For simplicity, the AACF of $a$ will be sometimes written as $R(a)(\tau)$.

Moreover, the periodic correlation function between $a$ and $b$ is defined as

$$ \theta(a, b)(\tau) \triangleq \sum_{t=0}^{N-1} a_t b_{t+\tau} = R(a, b)(\tau) + R^*(b, a)(N - \tau), \quad 0 \leq \tau \leq N - 1, $$

(2)

where the subscript $t + \tau$ is performed in modulo $N$. Similarly, $\theta(a, a)(\tau)$ [i.e., periodic auto-correlation function (PACF)] will be sometimes written as $\theta(a)(\tau)$.

Let $C = \{C^0, C^1, \ldots, C^{N-1}\}$ be a set of $K$ matrices, each of order $M \times N$, i.e.,

$$ C^\nu = \begin{bmatrix} e_0^\nu & e_1^\nu & \cdots & e_{M-1}^\nu \\ e_{0,0}^\nu & e_{0,1}^\nu & \cdots & e_{0,N-1}^\nu \\ \vdots & \vdots & \ddots & \vdots \\ e_{M-1,0}^\nu & e_{M-1,1}^\nu & \cdots & e_{M-1,N-1}^\nu \end{bmatrix} $$

(3)

in which every row sequence $e_m^\nu$ ($0 \leq \nu \leq K - 1, 0 \leq m \leq M - 1$) in $C^\nu$ has identical energy of $N$, i.e., $\|e_m^\nu\|_F^2 = N$.

Thus, each matrix in $C$ has identical energy of $MN$, i.e., $\|C^\nu\|_F^2 = MN$. Throughout this paper, the superscripts related to these $K$ matrices are used to label their indices in $C$ and have nothing to do with taking powers.

Define the “periodic correlation sum” and “aperiodic correlation sum” of matrices $C^\mu$ and $C^\nu$ as

$$ \theta(C^\mu, C^\nu)(\tau) \triangleq \sum_{m=0}^{M-1} \theta(e_m^\mu, e_m^\nu)(\tau), \quad 0 \leq \mu, \nu \leq K - 1, $$

(4)

and

$$ R(C^\mu, C^\nu)(\tau) \triangleq \sum_{m=0}^{M-1} R(e_m^\mu, e_m^\nu)(\tau), \quad 0 \leq \mu, \nu \leq K - 1, $$

(5)

respectively.
Definition 1: Define the periodic tolerance (also called the “maximum periodic correlation magnitude”) of \( C \) as

\[
\delta_{\text{max}} \triangleq \max \left\{ \left| \theta (C^\mu, C^\nu) (\tau) \right| : \mu = \nu, 0 < \tau \leq N - 1, \mu \neq \nu, 0 < \tau \leq N - 1 \right\}.
\]

\( C \) is called a periodic quasi-complementary sequence set (QCSS) if \( 0 < \delta_{\text{max}} \ll MN \) and a periodic perfect complementary sequence set (PCSS) if \( \delta_{\text{max}} = 0 \).

Definition 2: By replacing \( \theta (C^\mu, C^\nu) (\tau) \) in (6) with \( R(C^\mu, C^\nu) (\tau) \), we define the aperiodic tolerance of \( C \) as \( \tilde{\delta}_{\text{max}} \). \( C \) is called an aperiodic QCSS if \( 0 < \tilde{\delta}_{\text{max}} \ll MN \) and an aperiodic PCSS if \( \delta_{\text{max}} = 0 \).

Clearly, an aperiodic PCSS is also a periodic PCSS, but the converse may not be true.

Remark 1: The transmission of a PCSS or a QCSS requires a multi-channel communication system, hence CSSSs are referred to as multi-channel sequence sets in Welch’s paper [26]. If \( M = 1 \), \( C \) reduces to a conventional single-channel sequence set. In this case, \( \delta_{\text{max}} \) and \( \tilde{\delta}_{\text{max}} \) stand for periodic tolerance and aperiodic tolerance for single-channel sequence sets, respectively.

B. Spectrally-Constrained Sequences (SCS)

Consider a cognitive radio (or cognitive radar) network whose specific spectral band is divided into \( N \) carriers. Denote by \( M = [m_0, m_1, \cdots, m_{N-1}]^T \) the “carrier marking vector” which gives the active status of the \( N \) carriers. Specifically, the value of \( m_k (0 \leq k \leq N - 1) \) is set to 1 if the \( k \)-th carrier is available for SCS transmission; otherwise, \( m_k = 0 \), indicating that this carrier is occupied/reserved by another user and thus transmission in this carrier is not allowed (i.e., forbidden carrier). Denote by \( \Omega \) the positions of all the forbidden carriers, i.e., \( \Omega = \{ k | m_k = 0 \} \). In this paper, \( \Omega \) is also referred to as the “spectral constraint”.

Let

\[
\Omega = \{ i_0, i_1, \cdots, i_{n-1} \} \subseteq \{0, 1, 2, \cdots, N - 1\}
\]

where \( 0 \leq i_0 < i_1 < \cdots < i_{n-1} \leq N - 1 \), i.e., \( |\Omega| = n \). Also, let \( \Omega^- \) be

\[
\Omega^- = \{ 0, 1, 2, \cdots, N - 1 \} \setminus \Omega
\]

\[
= \{ j_0, j_1, \cdots, j_{N-1-n} \},
\]

with \( 0 \leq j_0 < j_1 < \cdots < j_{N-1-n} \leq N - 1 \). For any time-domain sequence \( c = [c_0, c_1, \cdots, c_{N-1}] \), consider its frequency-domain dual as \( D = \mathbf{c} F_N \). Throughout this paper, we consider constant-energy sequences with \( \|c\|^2_F = N \). It then follows from the Parseval’s theorem that \( \|D\|^2_F = \|c\|^2_F = N \).

Definition 3: For any row sequence \( c^\nu_m (0 \leq \nu \leq K - 1, 0 \leq m \leq M - 1) \) in \( C \) (see Section II.A), denote by \( D^\nu_m = [D^\nu_{m,0}, D^\nu_{m,1}, \cdots, D^\nu_{m,N-1}] \) its frequency-domain dual. The matrix set \( C \) is said to be a spectrally-constrained complementary sequence set (SC-CSS) if

\[
Z_{m,f} = \sum_{\nu=0}^{K-1} |D^\nu_{m,f}|^2 \leq \rho, \text{ for any } f \in \Omega, 0 \leq m \leq M - 1,
\]

where \( \rho \) is a small positive value satisfying \( \rho < K \).

Remark 2: The physical meaning of (9) arises from the uplink channel of a cognitive network, in which \( K \) users communicate over \( M \) non-interfering channels, each of which consisting of \( (N - n) \) available carriers specified by \( \Omega^- \). To suppress or minimize the interference incurred by these \( K \) users to the forbidden carriers, it requires that the total power received at the \( f \)-th carrier \((f \in \Omega)\) should be less than certain small amount. In this paper, \( \rho \) is called the “maximal power leakage” over the forbidden carriers. Ideally, we wish to have \( \rho = 0 \), implying zero interference incurred to the forbidden carriers.

Example 1: Let \( N = 15 \) and \( \Omega = \{3, 4, 10, 11\} \). The corresponding spectral map is shown in Fig. 1, where the blue solid arrows refer to the available carriers which can be used for SCS transmission and the red dashed arrows refer to the forbidden carriers. The magnitudes of the blue- and red-arrows stand for the maximal power levels that can be allocated to these individual carriers.

C. Cyclic Difference Set

For any subset \( D = \{d_0, d_1, \cdots, d_{n-1}\} \subseteq Z_N = \{0, 1, \cdots, N - 1\} \), the difference function of \( D \) is defined as

\[
d_D (\tau) = \| (\tau + D) \cap \Omega^- \|, \quad \tau \in Z_N.
\]

Definition 4: [30] \( D \) is said to be an \((N, n, \lambda)\) “cyclic difference set” if and only if \( d_D (\tau) \) takes on the value \( \lambda \) for \( N - 1 \) times when \( \tau \) ranges over the nonzero elements of \( Z_N \). Equivalently, \( D \) is also said to be an \((N, n, \lambda)\) “cyclic difference set” if and only if the \( n(n-1) \) differences

\[
(d_i - d_j) \mod N, \quad i \neq j
\]

take all possible nonzero values \( 1, 2, \cdots, N - 1 \), with each value appearing exactly \( \lambda \) times.

An example of cyclic difference set is given below.

Example 2: \( D = \{1, 2, 4\} \) is a \((7, 3, 1)\) cyclic difference set because the 6 differences of \( d_i - d_j \mod 7 \) (where \( i \neq j \)) take every value in \( \{1, 2, \cdots, 6\} \) once only, i.e.,

\[
\begin{align*}
1 - 2 &\equiv 6, \quad 1 - 4 \equiv 4, \quad 2 - 4 \equiv 5, \\
2 - 1 &\equiv 4, \quad 4 - 1 \equiv 3, \quad 4 - 2 \equiv 2.
\end{align*}
\]

It is noted that we have \( n(n-1) = (N-1)\lambda \). We present the following lemma which will be used in Subsection IV-B.

Lemma 1: [31] For an \((N, n, \lambda)\) cyclic difference set \( D \), we have

\[
\left| \sum_{i=0}^{n-1} \omega_N^{i \tau d_i} \right| = \sqrt{n - \lambda} = \sqrt{n(n-n) / (N-1)},
\]

(10)
where $\tau$ is an integer which satisfies $\tau \neq 0 \mod N$.

**Proof:** Consider

$$\sum_{i=0}^{n-1} \omega_N^{\tau d_i} = n + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \omega_N^{\tau (d_i - d_j)} = n \sum_{m=1}^{N-1} \omega_N^{\tau m} = -\lambda.$$  \hspace{1cm} (12)

for integer $\tau \neq 0 \mod N$. By (11) and (12), the proof of this lemma follows.

### III. CORRELATION LOWER BOUNDS OF SCSs

In this section, we derive the periodic- and aperiodic-correlation lower bounds for both single-channel SCSs and multi-channel SCSs (i.e., SC-CSSs). Since the correlation lower bounds of single-channel SCSs can be obtained from that of SC-CSSs by setting $M = 1$, our derivation will focus on the generic SC-CSSs in the sequel.

#### A. Periodic Correlation Lower Bound

Consider the SC-CSS $\mathcal{C}$ given in Definition 3. We express below the PACF of $c_m^\nu$, the $m$th row sequence of matrix $\nu$  

$$0 \leq \nu \leq K - 1, 0 \leq m \leq M - 1,$$

in the form of its frequency domain dual $D_m^\nu$.

$$\theta (c_m^\nu) (\tau) = \sum_{i=0}^{N-1} c_{m,i}^\nu (c_{m,i+\tau})^* \hspace{1cm} (13)$$

Thus,

$$\sum_{\nu=0}^{K-1} \theta (c_m^\nu) (\tau) = \sum_{f=0}^{N-1} \left| D_{m,f}^\nu \right|^2 \omega_N^{-\tau f}. \hspace{1cm} (14)$$

Let

$$\chi = \{ \mathbf{X}^0, \mathbf{X}^1, \cdots, \mathbf{X}^{M-1} \} \hspace{1cm} (15)$$

be a set of $M$ matrices (each of order $K \times N$) which is associated with the matrix set $\mathcal{C}$ in (3) as follows.

$$\mathbf{X}^m = \begin{bmatrix} c_0^m & c_1^m & \cdots & c_{K-1}^m \\ c_m^0 & c_m^1 & \cdots & c_m^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_m^{K-1} & c_m^{K-1} & \cdots & c_m^{K-1} \end{bmatrix}_{K \times N} \hspace{1cm} (16)$$

where $0 \leq m \leq M - 1$. By the definition of “periodic correlation sum” in (4), we have

$$\theta (\mathbf{X}^m) (\tau) = \sum_{\nu=0}^{K-1} \theta (c_m^\nu) (\tau). \hspace{1cm} (17)$$

Now, we recall $Z_{m,f}$ defined in (9). It is interesting to point out that

$$\sum_{f=0}^{N-1} Z_{m,f} = KN. \hspace{1cm} (18)$$
Moreover, it follows from (14) that
\[ \theta (X^m) (\tau) = \sum_{f=0}^{N-1} Z_{m,f} \omega_N^{-\tau f}. \] (19)

Let
\[ \Theta_m \triangleq [\theta (X^m) (0), \theta (X^m) (1), \ldots, \theta (X^m) (N-1)]^T, \]
\[ Z_{m} \triangleq [Z_{m,0}, Z_{m,1}, \ldots, Z_{m,N-1}]^T. \] (20)

By (19) and (20), we assert that \( \Theta_m = \sqrt{N} F_N Z_m. \) Next, we prove the following lemma on periodic QCSS.

**Lemma 2:**
\[ \sum_{m,m' = 0}^{M-1} \sum_{\tau = 0}^{N-1} \left| \theta (C^m, C^{m'}) (\tau) \right|^2 \geq \sum_{m=0}^{M-1} \sum_{\tau = 0}^{N-1} \sum_{\nu=0}^{K-1} \left| \theta (c_m^\nu, c_{m'}^\nu) (\tau) \right|^2. \] (21)

**Proof:** Since
\[ \sum_{\tau = 0}^{N-1} \left| \theta (C^m, C^{m}) (\tau) \right|^2 \]
\[ = \sum_{m,m' = 0}^{M-1} \sum_{\tau = 0}^{N-1} \sum_{t,t' = 0} c_{m,t}^\nu (c_{m',t'}^\nu)^* \left( \sum_{\tau = 0}^{N-1} c_{m,t+\tau}^\nu (c_{m',t'+\tau}^\nu)^* \right)^* \]
\[ = \sum_{m,m' = 0}^{M-1} \sum_{\tau = 0}^{N-1} \sum_{t,t' = 0} c_{m,t}^\nu (c_{m',t'}^\nu)^* \left( \sum_{\tau = 0}^{N-1} c_{m,t+\tau}^\nu (c_{m',t'+\tau}^\nu)^* \right)^* \]
\[ = \sum_{m,m' = 0}^{M-1} \sum_{\tau = 0}^{N-1} \theta (C^m, C^{m}) (\tau) \theta (C^{m'}, C^{m'}) (t-t'), \]
we have
\[ \sum_{m,m' = 0}^{M-1} \sum_{\tau = 0}^{N-1} \left| \theta (C^m, C^{m'}) (\tau) \right|^2 \]
\[ \geq \sum_{m,m' = 0}^{M-1} \sum_{\tau = 0}^{N-1} \sum_{\nu=0}^{K-1} \left| \theta (c_m^\nu, c_{m'}^\nu) (\tau) \right|^2 \]
\[ \geq \sum_{m,m' = 0}^{M-1} \sum_{\tau = 0}^{N-1} \sum_{\nu=0}^{K-1} \left| \theta (c_m^\nu, c_{m'}^\nu) (\tau) \right|^2, \]
where the equality is achieved if and only if
\[ \sum_{\nu=0}^{K-1} \theta (c_m^\nu, c_{m'}^\nu) (\tau) = 0, \forall m \neq m'. \] (24)

**Remark 3:** To meet lower bound of (21) with equality, (24) implies that
\[ \theta (X^m, X^{m'}) (\tau) = 0, \forall m \neq m'. \] (25)

Namely, the matrix set \( \chi \) (which is associated with \( C \)) in (15) should have zero periodic cross-correlation sums.

It is interesting to note the right-hand side (RHS) term of (21) can be expressed as
\[ \sum_{m=0}^{M-1} \sum_{\tau = 0}^{N-1} \left| \sum_{\nu=0}^{K-1} \theta (c_m^\nu) (\tau) \right|^2 = \sum_{m=0}^{M-1} \Theta_m^H \Theta_m. \] (26)

To lower bound the periodic tolerance of the SC-CSS \( C \), we consider the following optimization problem.

**Problem 1:**
\[ \min \Theta_m^H \Theta_m = N \cdot \min \sum_{f=0}^{N-1} Z_{m,f} Z_{m,f}, \] where \( 0 \leq m \leq M-1, \)
subject to (1) : \( \sum_{f=0}^{N-1} Z_{m,f} = K N, \ Z_{m,f} \geq 0; \)
(2) : \( 0 \leq Z_{m,f} \leq \rho \ll K, \forall f \in \Omega. \) (27)

To solve Problem 1, we first allocate \( Z_{m,f} \) with fixed values for all \( f \in \Omega. \) Specifically, we consider
\[ Z_{m,0} = \rho_0, Z_{m,i} = \rho_1, \ldots, Z_{m,2n-1} = \rho_{n-1}, \] (28)
with
\[ 0 < \rho_t \leq \rho, \forall t \in \{0, 1, \ldots, n-1\}. \]

In this case, we have
\[ \sum_{f=0}^{N-1} Z_{m,f} = K N - \sum_{t=0}^{n-1} \rho_t. \] (30)

Applying the Cauchy-Schwarz inequality, we obtain
\[ \sum_{f=0}^{N-1} Z_{m,f}^2 \geq \frac{(K N - \sum_{t=0}^{n-1} \rho_t)^2}{N-n} + \sum_{t=0}^{n-1} \rho_t. \] (31)

To further lower bound the RHS term of (31), let us define
\[ g(\rho_0, \rho_1, \ldots, \rho_{n-1}) = \frac{(K N - \sum_{t=0}^{n-1} \rho_t)^2}{N-n} + \sum_{t=0}^{n-1} \rho_t. \] (32)
Calculating the derivative of \( g(\rho_1, \rho_2, \cdots, \rho_n) \) with respect to \( \rho_i \) \((1 \leq i \leq n-1)\), we have
\[
\frac{\partial g}{\partial \rho_i} = -\frac{2}{N-n} \left( K^{\frac{n-1}{2}} \sum_{t=0}^{n-1} \rho_t \right) + 2\rho_i = 2\rho_i + \frac{2}{N-n} \sum_{t=0}^{n-1} \rho_t - \frac{2}{N-n} KN = 0. \tag{33}
\]
By noting \( \rho_i \leq \rho \leq K \), the derivative can be upper bounded as follows.
\[
\frac{\partial g}{\partial \rho_i} \leq 2\rho \left( 1 + \frac{n}{N-n} \right) - \frac{2}{N-n} KN = 2 \rho - K < 0. \tag{34}
\]
This shows that \( g(\rho_0, \rho_1, \cdots, \rho_{n-1}) \) is monotonically decreasing with respect to each variable \( \rho_i \). Therefore, the minimum value of \( g(\rho_0, \rho_1, \cdots, \rho_{n-1}) \) is achieved at
\[
\rho_0 = \rho_1 = \cdots = \rho_{n-1} = \rho. \tag{35}
\]
As a result, we have
\[
\sum_{m=0}^{m=M-1} \Theta_m^H \Theta_m \geq MN^2 - \frac{2KN\rho + n\rho^2}{N-n}, \tag{36}
\]
where the equality is met if and only if the following power allocation is applied for all \( 0 \leq m \leq M-1 \).
\[
Z_{m,f} = \left\{ \frac{KN-n\rho}{N-n}, \forall \ f \in \Omega \right\}. \tag{37}
\]
On the other hand, the left-hand side (LHS) term in (21) satisfies the following upper bound.
\[
\sum_{\mu,\nu=0}^{K-1} \sum_{\tau=0}^{N-1} \left| \langle C^\mu, C^\nu \rangle (\tau) \right|^2 \leq \left( K^2N - K \right) \delta_{\text{max}}^2 + KMN^2. \tag{38}
\]
By (38), (21), (26), and (36), we arrive at the following theorem.

**Theorem 1:** For the SC-CSS \( C \) defined in (9), the periodic tolerance of \( C \) satisfies the following lower bound.
\[
\delta_{\text{max}} \geq MN \cdot \sqrt{\frac{K \alpha}{K N - 1} + \frac{\beta}{K N - 1}}, \tag{39}
\]
where
\[
\alpha = \frac{N}{N-n}, \quad \beta = \frac{n\rho^2 - 2nK\rho}{(N-n)KM}. \tag{40}
\]

**Remark 4:** If \( n = 0, M \geq 2 \), the lower bound in (39) reduces to the Welch bound on the periodic tolerance of QCSS, i.e.,
\[
\delta_{\text{max}} \geq MN \cdot \sqrt{\frac{K}{K N - 1}}. \tag{41}
\]
In addition, if \( n = 0, M = 1 \), the lower bound in (39) reduces to the Welch bound for traditional (single-channel) sequences, i.e.,
\[
\delta_{\text{max}} \geq N \cdot \sqrt{\frac{K}{K - 1}}. \tag{42}
\]

**Corollary 1:** If \( n > 0, \rho = 0 \), the lower bound in (39) reduces to
\[
\delta_{\text{max}} \geq MN \cdot \sqrt{(K - M)N + nM \over (N-n)(K N - 1)M}. \tag{43}
\]
Moreover, setting \( M = 1 \) in (39), we obtain the periodic-correlation lower bound for traditional (single-channel) sequences as follows.
\[
\delta_{\text{max}} \geq N \cdot \sqrt{(K - 1)N + n \over (N-n)(K N - 1)}. \tag{44}
\]

**Remark 5:** It is interesting to note that (44) implies that perfect SCS set with zero non-trivial correlations does not exist if \( n > 0 \). On the other hand, perfect sequence set exists if and only if \( K = 1 \) and \( n = 0 \), i.e., all the carriers should be contiguous and allocated with identical power.

**B. Aperiodic Correlation Lower Bound**

For any \( 0 \leq \nu \leq K - 1 \), define \( C^\nu \triangleq C^\nu(0^{M \times N}) \), a matrix of order \( M \times 2N \) obtained from the horizontal concatenation of \( C^\nu \) and \( 0^{M \times N} \); Furthermore, define \( \bar{C}_m^\nu \triangleq \bar{C}_m^\nu(0^{1 \times N}) \). For every \( \bar{C}_m^\nu \), \( 0 \leq \nu \leq K - 1 \), \( 0 \leq m \leq M - 1 \), we have
\[
\bar{D}_{m,f}^\nu = \bar{C}_m^\nu Z_{2N}. \tag{45}
\]
Similar to the derivation for periodic correlation lower bound in Section III.A, let
\[
\bar{X} = \left\{ \bar{X}^0, \bar{X}^1, \cdots, \bar{X}^{M-1} \right\} \tag{46}
\]
be a set of \( M \) matrices (each of order \( K \times 2N \)) which is associated with the matrix set \( C \) in (3) as follows.
\[
\bar{X}^m = \begin{bmatrix} \bar{C}_m^0 \\ \vdots \\ \bar{C}_m^{K-1} \end{bmatrix} \quad , \quad 0 \leq m \leq M - 1. \tag{47}
\]
Define \( \bar{Z}_{m,f} \triangleq \sum_{\nu=0}^{K-1} |\bar{D}_{m,f}^\nu|^2 \) and
\[
\bar{Z}_m \triangleq \left[ \theta \left( \bar{X}^m \right)(0), \theta \left( \bar{X}^m \right)(1), \cdots, \theta \left( \bar{X}^m \right)(2N - 1) \right]^T, \tag{48}
\]
By (47), we assert that the identity in (45) can be rewritten as
\[
\theta \left( \bar{X}^m \right)(\tau) = \sum_{f=0}^{2N-1} \bar{Z}_{m,f} \omega_{2N}^{-\tau f}. \tag{49}
\]
Furthermore, define
\[
\bar{D}^\nu_{m,even} \triangleq \bar{D}^\nu_{m,0}, \bar{D}^\nu_{m,2}, \cdots, \bar{D}^\nu_{m,2N-2}, \tag{50}
\]
\[
\bar{D}^\nu_{m,odd} \triangleq \bar{D}^\nu_{m,1}, \bar{D}^\nu_{m,3}, \cdots, \bar{D}^\nu_{m,2N-1}. \tag{51}
\]
It can be readily shown that

\[ \bar{D}_{m,\text{even}}^\nu = \frac{1}{\sqrt{2}} D_{m,\text{odd}}^\nu, \]

\[ \|\bar{D}_{m,\text{even}}^\nu\|_F = \|D_{m,\text{odd}}^\nu\|_F = \frac{N}{2}, \]

where \( D_{m,\text{odd}}^\nu \) is the frequency-domain dual of \( c_{m,\text{odd}}^\nu \) (see Section III.A). The spectral constraint imposed can now be translated to

\[ \bar{Z}_{m,2f} \leq \frac{\rho}{2} \ll \frac{K}{2}, \forall f \in \Omega. \]

Clearly, \( \bar{\Theta}_m = \sqrt{2N}F_{2N} \bar{Z}_m \).

Similar to Lemma 2, we need the following lemma on aperiodic QCSS.

**Lemma 3:**

\[
\sum_{\mu, \nu = 0}^{K-1} \sum_{\tau = 1-N}^{N-1} |R(Cm, C\nu) (\tau)|^2 \geq \sum_{m=0}^{M-1} \sum_{\tau = 0}^{N-1} |\sum_{\nu=0}^{K-1} R\left(c_{m}^\nu (\tau)\right)|^2.
\]

The proof of this lemma follows by noting the following two identities.

\[
\sum_{\tau = 1-N}^{N-1} |R(Cm, C\nu) (\tau)|^2 = \sum_{\tau = 0}^{N-1} |\sum_{\nu=0}^{K-1} R\left(c_{m}^\nu (\tau)\right)|^2,
\]

which is also useful.

**Remark 6:** To meet the lower bound of (53) with equality, it requires that

\[ \theta\left(\bar{X}_m, \bar{X}_m^{m'}\right) (\tau) = 0, \forall m \neq m'. \]

Namely, the matrix set \( \bar{X} \) (which is associated with \( C \)) in (46) should have zero periodic cross-correlation sums. This is equivalent to the following statement: the matrix set \( \chi \) in (15) should have zero aperiodic cross-correlation sums.

By noting that \( \theta\left(c_{m}^\nu\right) (N) = 0 \) for any \( 0 \leq \nu \leq K-1, 0 \leq m \leq M-1 \), the RHS term of (53) can be expressed as

\[
\sum_{m=0}^{M-1} \sum_{\tau = 1-N}^{N-1} \sum_{\nu=0}^{K-1} R\left(c_{m}^\nu (\tau)\right) ^2 = \sum_{m=0}^{M-1} \sum_{\tau = 0}^{N-1} \sum_{\nu=0}^{K-1} \theta\left(c_{m}^\nu (\tau)\right) ^2
\]

\[ = \sum_{m=0}^{M-1} \bar{\Theta}_m^{H} \bar{\Theta}_m. \]

To lower bound the aperiodic tolerance of the SC-CSS \( C \) specified in (9), we consider the following optimization problem.

**Problem 2:**

\[ \min \bar{\Theta}_m^{H} \bar{\Theta}_m = 2N \cdot \min \bar{Z}_m^{H} \bar{Z}_m, \text{ where } 0 \leq m \leq M-1, \]

subject to

\[ \sum_{f=0}^{2N-1} \bar{Z}_{m,f} = KN, \bar{Z}_{m,f} \geq 0; \]

\[ \sum_{f=0}^{2N-1} \leq \frac{\rho}{2} \ll \frac{K}{2}, \forall f \in \Omega. \]

By applying the optimization technique similar to that used for Problem 1, we have

\[ \bar{Z}_m^{H} \bar{Z}_m \geq \frac{(KN/2 - n\rho/2)^2}{N - n} + n(\rho/2)^2 + \frac{(KN/2)^2}{N}. \]

Consequently,

\[ \sum_{m=0}^{M-1} \bar{\Theta}_m^{H} \bar{\Theta}_m \geq MN^2 \cdot \frac{2K^2N - 2Kn\rho - K^2n + n\rho^2}{2(N - n)}, \]

where the equality is achieved if and only if the following power allocation is applied.

\[ \bar{Z}_{m,f} = \begin{cases} \frac{K}{2} \cdot \nu \left( \frac{KN/2 - n\rho/2}{2(N - n)} \right), & \forall \text{ odd } f; \\ \frac{\rho}{2}, & \forall \text{ even } f \text{ with } f \in \Omega; \end{cases} \]

On the other hand, the LHS term in (53) satisfies the following upper bound.

\[ \sum_{\mu, \nu = 0}^{K-1} \sum_{\tau = 1-N}^{N-1} |R(Cm, C\nu) (\tau)|^2 \leq K \left(2N - 1\right) K - 1 \bar{\delta}_{\max}^2 + KM^2N^2. \]

By (62), (53), (57), and (60), we obtain the second theorem of this paper.

**Theorem 2:** For the SC-CSS \( C \) defined in (9), the aperiodic tolerance of \( C \) satisfy the following lower bound.

\[ \bar{\delta}_{\max} \geq MN \cdot \sqrt{\frac{\frac{K^2}{4} \alpha - 1}{(2N - 1) K - 1} + \frac{\beta}{K(2N - 1) - 1}}, \]

where

\[ \alpha = \frac{N}{N - n}, \quad \beta = \frac{n\rho^2 - 2nK\rho - nK^2}{2(N - n)KM}. \]

**Remark 7:** If \( n = 0, M \geq 2 \), the lower bound in (63) reduces to the Welch bound on aperiodic tolerance of QCSS, i.e.,

\[ \bar{\delta}_{\max} \geq MN \cdot \sqrt{\frac{K - 1}{K(2N - 1) - 1}}. \]

In addition, if \( n = 0, M = 1 \), the lower bound in (63) reduces to the Welch bound for traditional (single-channel) sequences, i.e.,

\[ \bar{\delta}_{\max} \geq N \cdot \sqrt{\frac{K - 1}{K(2N - 1) - 1}}. \]
Corollary 2: If \( n > 0, \rho = 0 \), the lower bound in (63) reduces to
\[
\bar{\delta}_{\text{max}} \geq M N \cdot \sqrt{\frac{K}{N} \frac{K - 1}{K(2N - 1)}},
\]
where \( \gamma = \frac{N - n}{N - n} \).

IV. DISCUSSIONS AND CONSTRUCTIONS

In this section, we shall first discuss tightness of the bounds by comparing with the results in [32]. Next, we shall consider the design of optimal unimodular single-channel SCSSs with uniformly low PACFs. These optimal SCSSs have potential applications such as synchronization and channel estimation in cognitive networks.

A. Tightness of The Bounds

We note that the periodic correlation lower bound in Theorem 1 can be rewritten as
\[
\delta_{\text{max}} \geq M N \cdot \sqrt{\frac{K - 1}{K(2N - 1)}} + \frac{n(K - \rho)^2}{(N - n)(K N - 1) K M}. \tag{68}
\]
Also, the aperiodic correlation lower bound in Theorem 2 can be rewritten as (68). From (68) and (69), we can readily see that both the correlation lower bounds in Theorems 1 and 2 are strictly tighter than the corresponding Welch bounds in (41) and (65), respectively for non-trivial SC-CSSs with \( n > 0 \) and \( \rho \ll K \).

We are aware that a collection of correlation lower bounds have been derived in [32] on SC-CSSs with low correlation zone (LCZ). By setting the value of LCZ \( L_{cz} \) to be \( L \) (which denotes the sequence length in [32]) in the periodic correlation lower bound in [32, (32)], we can show that it reduces to (43) in Section IIIA. Of this paper. Nevertheless, all the correlation lower bounds in [32] (including [32, (32)]) only apply to SC-CSSs with zero power leakage (i.e., \( \rho = 0 \)), in contrast to our proposed lower bounds in Theorems 1 and 2 which are applicable for positive power leakages.

For the aperiodic correlation lower bound in [32, (33)], we obtain the following bound by setting \( L_{cz} = L \):
\[
\bar{\delta}_{\text{max}} \geq M N \cdot \sqrt{\frac{K - 1}{K(2N - 1)}}, \tag{70}
\]
By some simple calculations, we can show that our proposed lower bound in (67) is tighter than that in (70).

B. Analytical Constructions of Optimal Single-Channel SCSSs

In general, analytical constructions (as opposed to numerical constructions) of generic SC-CSSs under arbitrary spectral constraints is a challenging task. However, we will show in this subsection that under certain conditions, it is possible to construct unimodular single-channel SCSSs (i.e., \( M = 1 \)) each having uniformly low PACF meeting the following lower bound [obtained by setting \( K = 1 \) in (44)] with equality.
\[
\delta_{\text{max}} \geq \frac{n(K - \rho)}{(N - n)(N - 1)}, \tag{71}
\]
Formally, our design task can be cast as follows.

Problem 3: For a given number of forbidden carriers \( n \), how to select a spectral constraint \( \Omega \) and then construct an unimodular SCSS \( \mathbf{c} = [c_0, c_1, \ldots, c_{N-1}] \) which has
\[
|\theta(\mathbf{c})(\tau)| = N \cdot \sqrt{\frac{n}{(N - n)(N - 1)}}, \quad \forall \tau \in \{1, 2, \ldots, N - 1\}. \tag{72}
\]

In this paper, solutions to Problem 3 are said to be optimal SCSSs. Recall that the frequency domain dual of \( \mathbf{c} \) is denoted by \( \mathbf{D} = [D_0, D_1, \ldots, D_{N-1}] \) (see Section II.B). To solve Problem 3, let
\[
\Theta_\mathbf{D} \triangleq |\theta(\mathbf{D})(0), \theta(\mathbf{D})(1), \ldots, \theta(\mathbf{D})(N - 1)|^T, \tag{73}
\]
\[
\zeta \triangleq [c_0^2, c_1^2, \ldots, c_{N-1}^2]^T.
\]

Similar to \( \Theta_m = \sqrt{N} F_N Z_m \) shown in Section IIIA, we have
\[
\Theta_\mathbf{D} = \sqrt{N} F_N^T \cdot \zeta, \tag{74}
\]
leading to \( \zeta = \frac{1}{\sqrt{N}} \cdot F_N \cdot \Theta_\mathbf{D} \). Note that \( \theta(\mathbf{D})(0) = \|D\|^2_P = N \). Thus, we have the following remark.

Remark 8: \( \zeta \) is an all-one vector (i.e., \( \mathbf{c} \) is unimodular) provided that \( \theta(\mathbf{D})(\tau) = 0 \) for all \( \tau \in \{1, 2, \ldots, N - 1\} \), i.e., the frequency domain dual \( \mathbf{D} \) is a perfect sequence with zero PACF sidelobes.

On the other hand, since (71) implies \( \rho = 0 \), the PACF \( \theta_{\mathbf{c}}(\tau) \) [see (13)] can be written as
\[
\theta(\mathbf{c})(\tau) = \sum_{f \in \Omega} |D_f|^2 \omega_N^{-\tau f}. \tag{75}
\]
To meet the lower bound in (71) with equality, it is necessary by (37) that
\[
Z_f = |D_f|^2 = \frac{N}{N - n}, \quad \forall f \in \Omega. \tag{76}
\]
Therefore, to construct an optimal SCSS with uniformly low PACF sidelobes, the following identity needs to be satisfied.
\[
\left| \sum_{f \in \Omega} \omega_N^{-\tau f} \right| = \frac{n(N - n)}{N - 1}, \tag{77}
\]
for all \( \tau \in \{1, 2, \ldots, N - 1\} \). Recalling Lemma 1, the equality of (78) is achieved provided that \( \Omega \) is an \( (N, N - n, \lambda) \) cyclic difference set with
\[
\lambda = \frac{(N - n)(N - n - 1)}{N - 1}.
\]
Equivalently, it requires that \( \Omega \) (i.e., the spectral constraint) is an \( (N, n, \lambda) \) cyclic difference set with \( \lambda = n(n - 1)/(N - 1) \).
We are now in a position to present the third theorem of this paper.

**Theorem 3:** An optimal SCS \( c \) with uniformly low PACF sidelobes is constructed if the following two conditions are satisfied: 1), the corresponding frequency-domain dual \( D \) has zero PACF sidelobes; 2), \( \Omega \) is an \((N, n, \lambda)\) cyclic difference set.

Next, we present two constructions of optimal SCSs from the theory of maximal-length shift register sequences.

**Construction 1:** Consider a maximal-length shift register sequence (i.e., m-sequence) \( \{b_t\} \) characterized by a primitive polynomial \( g(x) = g_0 x^k + g_1 x^{k-1} + \cdots + g_k, g_k \in \text{GF}(q) \), where \( q \) is an odd prime power. Let \( \alpha \) be a primitive element of \( \text{GF}(q) \). For parameter \( N = \frac{q^k - 1}{q - 1} \), define a ternary sequence \( D = [D_0, D_1, \ldots, D_{N-1}] \) as follows.

\[
D_f = \begin{cases} 
0, & \text{if } b_f = 0; \\
(-1)^f \cdot (-1)^{\mu}, & \text{if } b_f = \alpha^{\mu}.
\end{cases}
\]

Such a ternary sequence \( D \) is an Ipatov sequence with zero PACF sidelobes [33]. Also, the Gordon-Mills-Welch (GMW) construction shows that \( \{f : b_f = 0, 0 \leq f \leq N - 1\} \) forms a \((\frac{q^k - 1}{q - 1}, \frac{q^{k-1} - 1}{q - 1}, \frac{q^{k-2} - 1}{q - 1})\) Singer difference set [34]. Therefore, the IDFT of \( \sqrt{\frac{N}{N-n}} \) is an optimal SCS, where the factor \( \sqrt{\frac{N}{N-n}} \) is to normalize the sequence energy to \( N \).

**Construction 2:** Let \( \{x_t\} \) and \( \{y_t\} \) be two m-sequences over \( \text{GF}(2) \) of length \( 2^{k+1} \) and \( 2^{k} \) (odd) satisfying \( y_t = x_{dt \mod 2^{k+1}} - 1 \) for \( t \in \{0, 1, \ldots, 2^k - 1\} \) (i.e., \( y_t \) is the d-decimation of \( \{x_t\} \)). Denote by \( \theta(x, y)(\tau) \) the periodic cross-correlation function between \( \{x_t\} \) and \( \{y_t\} \). Suppose \( k \) and \( c \) are co-prime, i.e., \( \gcd(k, c) = 1 \). If \( d = 2^c + 1 \) or \( d = 2^{2c} - 2^c + 1 \), then as \( \tau \) ranges from 0 to \( 2^k - 2 \), \( \theta(x, y)(\tau) \) takes on the three values below.

\[
\begin{align*}
-1 + 2^{(k+1)/2} & \quad \text{occurs } 2^{k-2} + 2^{(k-3)/2} \text{ times}; \\
-1 & \quad \text{occurs } 2^{k-1} \text{ times}; \\
-1 - 2^{(k+1)/2} & \quad \text{occurs } 2^{k-2} - 2^{(k-3)/2} \text{ times}.
\end{align*}
\]

Define a ternary sequence \( D = [D_0, D_1, \ldots, D_{N-1}] \) as follows.

\[
D_\tau = \begin{cases} 
1, & \text{if } \theta(x, y)(\tau) = -1 + 2^{(k+1)/2}; \\
0, & \text{if } \theta(x, y)(\tau) = -1; \\
-1, & \text{if } \theta(x, y)(\tau) = -1 - 2^{(k+1)/2}.
\end{cases}
\]

Ternary sequence \( D \) is a Shedd-Sarwate sequence with zero PACF sidelobes [35]. It has been shown by Gold [36] and Kasami [37] that \( \{ \tau : \theta(x, y)(\tau) = -1, 0 \leq \tau \leq N - 1 \} \) equals to the position set of all the 1’s of another m-sequence (with identical length of \( 2^k - 1 \)) and hence it forms a \((2^k - 1, 2^{k-1} - 1, 2^{k-2} - 1)\) Singer difference set. Therefore, the IDFT of \( D \cdot \sqrt{\frac{N}{N-n}} \) is an optimal SCS.

**Example 3:** Consider an m-sequence pair \( x \) and \( y \), having identical length of 31 (i.e., \( k = 5 \)), as shown in (82). Note that the decimation factor \( d = 2^{2c} - 2^c + 1 = 57 \) for \( c = 3 \), and hence stand for 1 and \(-1\), respectively. One can easily show that \( D_1 \) in (83) is a ternary sequence with zero PACF sidelobes. Note that

\[
\Omega_1 = \{0, 1, 5, 6, 7, 8, 11, 13, 15, 16, 18, 19, 20, 22, 28\},
\]

which is a \((31, 15, 7)\) Singer difference set. Applying the IDFT to \( D_1 \cdot \sqrt{\frac{31}{16}} \), we obtain the optimal unimodular SCS \( c_1 \) in Table I. It is stressed that \( c_1 \) can be applied to a cognitive radio/radar system consisting of 31 carriers, where forbidden carriers are specified in \( \Omega_1 \).

As a comparison, we consider a random (frequency-domain) ternary sequence (also of length 31) as follows.

\[
D_2 = [0000+---+---+---0000+---+---+---0000].
\]

The spectral constraint of \( D_2 \) is

\[
\Omega_2 = \{1, 2, 3, 4, 13, 14, 15, 16, 20, 21, 22, 23, 29, 30, 31\}.
\]

Note that \( \Omega_2 \) has identical size with \( \Omega_1 \), i.e., both have identical number of forbidden carriers. Similarly, we apply the IDFT to \( D_2 \cdot \sqrt{\frac{31}{16}} \) and obtain the corresponding random (time-domain) SCS \( c_2 \) in Table I. However, \( \Omega_2 \) is not a cyclic difference set and hence Condition 2 of Theorem 3 cannot be met.

To visualize both sequences, we plot their (individual) frequency-domain magnitudes, time-domain magnitudes, and their PACFs of time-domain sequences in Fig. 2. It is shown that the PACF sidelobes of \( c_1 \) (see Fig. 2-a) have identical magnitude of \( N \cdot \sqrt{\frac{n}{(N-n)(N-1)}} \approx 5.4801 \), whereas that of \( c_2 \) vary as the time-shift \( \tau \) changes (see Fig. 2-b). Also, the optimal sequence \( c_1 \) has identical magnitude of 1, in contrast to \( c_2 \) with large fluctuations in magnitudes (i.e., higher PAPR).

We present the following corollary for new optimal SCSs from an existing optimal one.

**Corollary 3:** Let \( c \) be an optimal SCS obtained from Construction 1 or Construction 2, where the corresponding frequency-domain dual and spectral constraint are \( D = [D_0, D_1, \ldots, D_{N-1}] \) and \( \Omega = \{i_0, i_1, \ldots, i_{n-1}\} \), respectively. Each of the following operations on \( D \) followed by the (normalized) IDFT will produce another optimal SCS.

- **reverse:** \( D = [D_{N-1}, D_{N-2}, \ldots, D_0] \),
- **cyclic-shift:** \( D = [D_{d+0}, D_{d+1}, \ldots, D_{d+(N-1)}] \),
- **decimation:** \( D = [D_{d+0}, D_{d+1}, \ldots, D_{d+(N-1)}] \),

where \( \gcd(d, N) = 1 \).
where their respective spectral constraints $\Omega'$ are: \(\{N - i_0, N - i_1, \ldots, N - i_{n-1}\}\), \(\{i_0 + d, i_1 + d, \ldots, i_{n-1} + d\}\), and \(\{i_0d, i_1d, \ldots, i_{n-1}d\}\). Here all the subscript indices in (84a)-(84d) as well as the digits in $\Omega'$ are obtained in modulo $N$ and $d$ is an integer.

**Proof:** We only prove the “decimation” case as the proof for the other two is straightforward. It is known that decimation of any periodic sequence does not change the maximal PACF sidelobe provided that the decimation factor is co-prime to the sequence length $N$ [38, page 195]. Hence, a decimated version of \(D\) with zero PACF sidelobes generates another perfect sequence. On the other hand, the spectral constraint $\Omega$ (a Singer difference set) in Construction 1 (or Construction 2) equals to the position set of all the 0’s (or 1’s) of an m-sequence. Also, (non-trivial) decimation of an m-sequence produces another m-sequence. Thus, the resultant spectral constraint (after decimation of \(D\)) $d \cdot \Omega$ will still be a Singer difference set. This shows that both conditions in Theorem 3 are satisfied after decimation and therefore, the (normalized) IDFT of \(D \cdot \sqrt{\frac{N}{N-n}}\) will also be an optimal SCS.

---

**TABLE I**

**Optimal SCS \(c_1\) and Random SCS \(c_2\), where \(i = \sqrt{-1}\).**

| \(c_1\) | 1.0000, -0.9327 + 0.3607i, 0.0143 + 0.9999i, 0.3624 + 0.9326i, 0.6405 - 0.7686i, -0.1525 - 0.9883i, -0.7692 - 0.6391i, 0.7006 - 0.7136i, 0.8561 + 0.5168i, 0.9538 - 0.3094i, 0.0495 - 0.9988i, 0.4394 + 0.8983i, -0.9492 + 0.3147i, 0.2517 + 0.9678i, -0.9931 - 0.1172i, -0.9716 + 0.2368i, -0.9716 - 0.2368i, -0.9931 + 0.1172i, 0.2517 - 0.9678i, -0.9492 - 0.3147i, 0.4394 - 0.8983i, 0.0495 + 0.9988i, 0.9538 - 0.3094i, 0.7006 + 0.7136i, -0.7692 + 0.6391i, -0.1525 + 0.9883i, 0.3624 - 0.9326i, 0.6405 + 0.7686i, 0.0143 - 0.9999i, -0.9327 - 0.3607i, |
| \(c_2\) | -0.5000, 0.4694 - 0.7588i, 0.2064 + 0.2297i, 0.0137 + 0.5956i, -0.9856 + 0.0528i, 0.8926 + 0.1697i, -0.6791 - 0.2544i, -0.7387 - 0.1965i, 0.6546 - 0.7073i, 1.4840 + 1.5829i, 0.3876 - 0.4539i, -1.2424 - 0.3278i, -1.0318 + 0.2068i, -0.5275 - 0.1136i, 1.3512 + 0.5189i, -0.0044 + 0.2851i, -0.0044 - 0.2851i, 1.3512 - 0.5189i, -0.5275 + 0.1136i, -1.0318 + 0.2068i, -1.2424 + 0.3278i, 0.3876 + 0.4539i, 1.4840 - 1.5829i, 0.6546 + 0.7073i, -0.7387 - 0.1965i, -0.6791 + 0.2544i, 0.8926 - 0.1697i, -0.9856 - 0.0528i, 0.0137 - 0.5956i, 0.2064 + 0.2297i, 0.4694 + 0.7588i, |

Fig. 2. Comparison of optimal SCS $c_1$ (shown in the left sub-figure) and random SCS $c_2$ (the right sub-figure) in Example 2, where the maximal power leakage $\rho = 0$. 

\[ x = [+---+-+-+-+++++---++---+-+-++-], \]
\[ y = [+----+-+-+++++---++---+-+-+++] , \]
\[ D_1 = \frac{\theta(x, y)(\tau) + 1}{8} \]
\[ = [00 - ++0000 +0 +0 +0 +0 + + + --0 +] , \]

\[ \theta(c_1)(\tau) = \frac{\theta(x, y)(\tau) + 1}{8} \]
\[ \theta(c_2)(\tau) = \frac{\theta(x, y)(\tau) + 1}{8} \]
Another possible sequence operation for optimal SCSs is the linear phase transform which is defined as
\[
\hat{D} = \left[ \omega_N^0 D_0, \omega_N^{1d} D_1, \ldots, \omega_N^{(N-1)d} D_{N-1} \right].
\]
However, the resultant spectral constraint will be exactly the same as \(\Omega\). It would be interesting to find new nontrivial transforms or permutations to \(D\) (similar to that in Corollary 3) such that many more different spectral constraints can be supported. Next, we present the following example on the application of “decimation” operation proposed in Corollary 3.

**Example 4:** Consider the optimal SCS \(e_1\) in Example 4, with frequency-domain dual \(D_1\) and spectral constraint \(\Omega_1\). Let \(d = 5\) be the decimation factor. Applying the decimation to \(D_1\), we obtain the following perfect (frequency-domain) ternary sequence
\[
\hat{D}_1 = [00-00++++0+-+000+0+-0++0-000+-],
\]
and the associated Singer difference set below.
\[
\hat{\Omega}_1 = \{0,1,3,4,10,14,15,16,18,20,23,25,26,27,28\}.
\]
It follows from Theorem 3 that the IDFT of \(\hat{D}_1\) is also an optimal SCS.

**Remark 9:** For any arbitrary optimal SCS not generated by Construction 1 and Construction 2, although sequence operations in (84a)-(84b) are applicable for new optimal SCSs, the same may not be held for “decimation” operation in (84d) because \(d \cdot \Omega\) may not necessarily be a cyclic difference set [39, Chapter III].

Finally, it should be pointed out that the proposed constructions for optimal SCS may not be feasible when a cyclic difference set cannot be found over the available non-contiguous carriers [41]. In this case, sequence synthesis based on numerical optimization will be more effective. Examples of such numerical algorithms can be found in [13], [40], [18], [19], [21]–[24].

V. CONCLUSIONS AND FUTURE DIRECTIONS

Spectrally-constrained sequences (SCSs) refer to sequences working over certain non-contiguous carriers. Unlike traditional sequences with no spectral constraints imposed in the frequency domain, SCSs are specifically designed to support spectrally-constrained systems such as overlay cognitive radio/radar networks.

In this paper, we have derived periodic- and aperiodic-correlation lower bounds (see Theorems 1 and 2 in Section III) for both multi-channel SCSs and single-channel SCSs. Each of these bounds is a function of the maximal power leakage (i.e., the maximal power leaking over the forbidden carriers) and reduces to one of the Welch bounds if the number of forbidden carriers is set to zero. We have also constructed two classes of unimodular single-channel SCSs with uniformly low PACF sidelobes meeting the lower bound (71) with equality. We have shown in Theorem 3 that such an optimal SCS is built provided: (1) the frequency domain dual of such SCS has zero PACF sidelobes and (2) its spectral constraint (denoted by \(\Omega\)) forms a cyclic difference set. The second condition of Theorem 3 has some practical significance in that given a set of non-contiguous carriers, those whose indices form (or tend to form) a cyclic different set will be able to construct an optimal (or near optimal) SCS.

A future task of this research is to construct optimal SCSs which have uniformly low auto- and cross-correlations approaching the derived bound set in Section III, numerically or analytically. It is also interesting to know whether these derived correlation bounds can be further tightening using techniques in [28], [42], [43] and [44]. It may also be interesting to apply SCSs to support carrier aggregation (CA), an emerging technique for substantial bandwidth expansion over fragmented spectrum which is one of the main features of LTE-Advanced (Release 10). Specifically, CA can support peak data rates by dynamically aggregating several contiguous or non-contiguous component carriers (CC) such as those unused scattered frequency bands and/or those allocated for some legacy systems (e.g., GSM and 3G systems) [45]–[47]. Although CA was limited to a few CC in its original proposal (for bandwidth up to 100 MHz), it has quickly evolved to massive CA for communications over a large number of non-contiguous carriers [48], [49]. In this application scenario, SCSs may be used as training sequences for synchronization and channel estimation or as spreading sequences for low-rate but ultra-reliable control signaling.

REFERENCES


